

Chapter V: Applications of Integration

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January, 2012

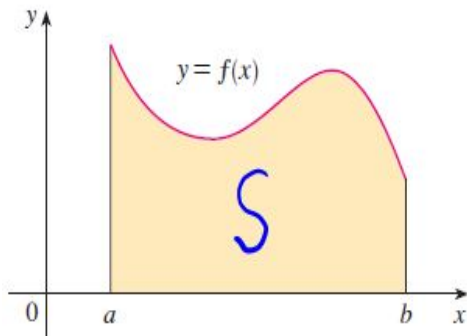
5.1 Areas between curves

Suppose f is continuous on $[a, b]$. Let

$$S := \{(x, y) : a \leq x \leq b, \quad 0 \leq y \leq f(x)\}.$$

Recall that if $f(x) \geq 0$ for any $x \in [a, b]$ then

$$\int_a^b f(x) dx = \text{the area of } S.$$



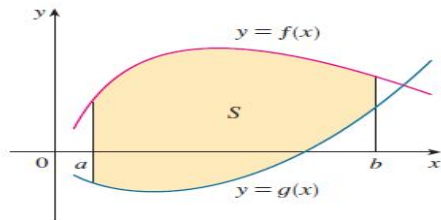
5.1 Areas between curves

1) Case of $f(x) \geq g(x)$ for $x \in [a, b]$

Suppose f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$ for $x \in [a, b]$.

Define

$$S = \{(x, y) : a \leq x \leq b; g(x) \leq y \leq f(x)\}.$$



Question: How to evaluate the area of S ?

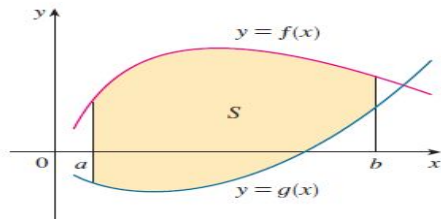
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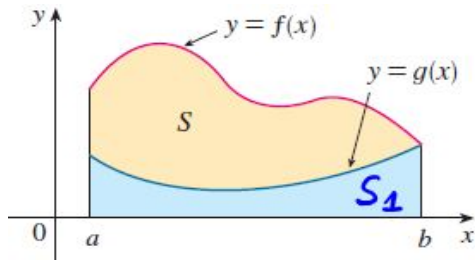
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Answer:

$$\text{Area of } S = \int_a^b [f(x) - g(x)] dx.$$

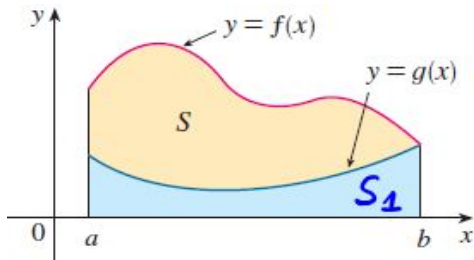
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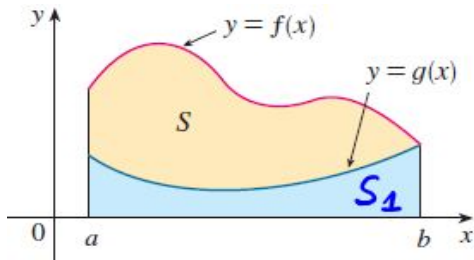


Since $f(x) \geq 0, x \in [a, b]$, we have

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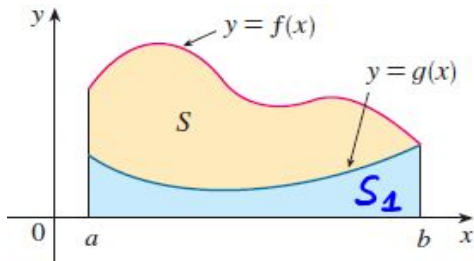


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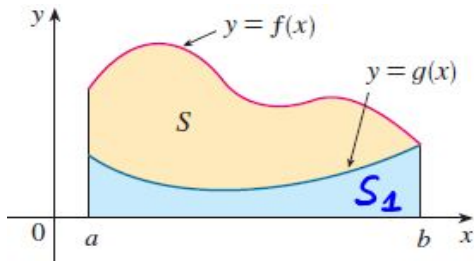
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$\int_a^b g(x)dx$ = the area of S_1 . Thus,

the area of S = (the area of $S \cup S_1$) - the area of S_1

$$= \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx.$$

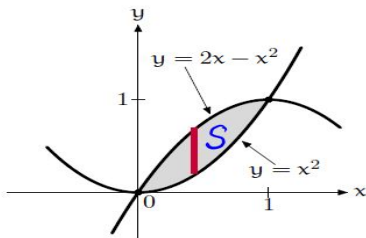
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This gives, $x = 0, y = 0$ and $x = 1, y = 1$.

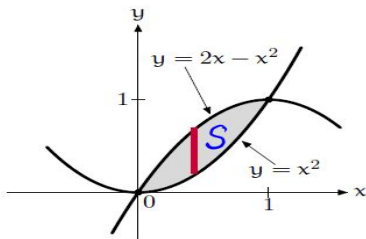


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Note that $2x - x^2 \geq x^2$, $x \in [0, 1]$. Therefore,

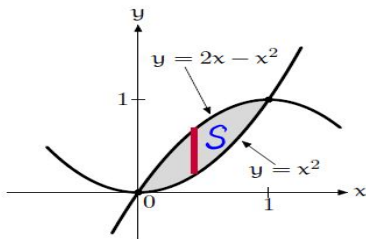
$$S = \{(x, y) : 0 \leq x \leq 1; x^2 \leq y \leq 2x - x^2\}.$$

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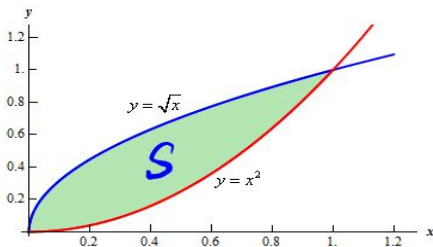
$$\text{the area of } S = \int_0^1 [(2x - x^2) - x^2] dx = \frac{1}{3}.$$

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Solution: The curves $y = x^2$ and $y = \sqrt{x}$ intersect at $x = 0$ and $x = 1$. Moreover, we have

$$\sqrt{x} \geq x^2, \quad \forall x \in [0, 1].$$

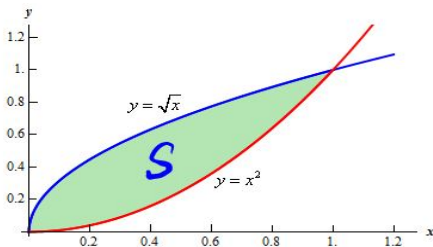


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Thus, the area of S is given by

$$\text{the area of } S = \int_0^1 (\sqrt{x} - x^2) dx = \left. \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right|_0^1 = \frac{1}{3}.$$

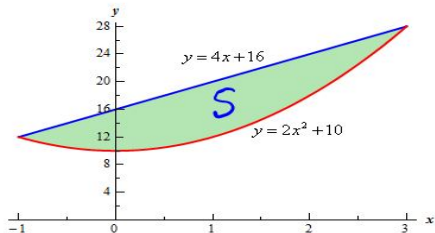
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Solution: We first find the points of intersection of the parabola $y = 2x^2 + 10$ and the line $y = 4x + 16$. Solving the system

$$y = 2x^2 + 10; \quad y = 4x + 16$$

gives $x = -1, y = 12$ and $x = 3, y = 28$.



Note that $(4x + 16) \geq (2x^2 + 10)$, $x \in [-1, 3]$.

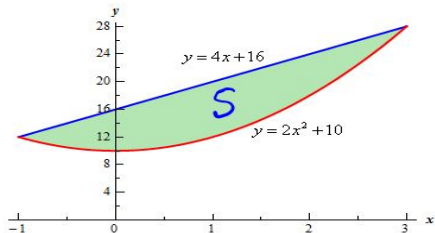
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$$S = \{(x, y) : -1 \leq x \leq 3; 2x^2 + 10 \leq y \leq 4x + 16\}.$$

Thus, the area of S is given by

$$\text{the area of } S = \int_{-1}^3 [(4x + 16) - (2x^2 + 10)] dx = \frac{64}{3}.$$

5.1 AREAS BETWEEN CURVES

5.1.1 AREA BETWEEN TWO CURVES

$y = f(x)$ AND $y = g(x)$

2) General Case

More generally,

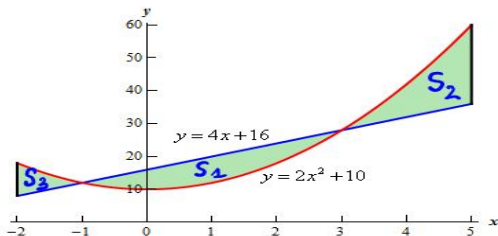
If f and g are continuous on $[a, b]$, then the area A between the the curves $y = f(x)$ and $y = g(x)$ over the interval is given by

$$A = \int_a^b |f(x) - g(x)| dx$$

Example: Determine the area of the region S bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.

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Solution: Here is the graph of S :



$$S = S_1 \cup S_2 \cup S_3$$

$$S_1 = \{(x, y) : -1 \leq x \leq 3; 2x^2 + 10 \leq y \leq 4x + 16\}.$$

$$S_2 = \{(x, y) : 3 \leq x \leq 5; 4x + 16 \leq y \leq 2x^2 + 10\}.$$

$$S_3 = \{(x, y) : -2 \leq x \leq -1; 4x + 16 \leq y \leq 2x^2 + 10\}.$$

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The area of S is given by

$$\int_{-2}^5 \left| (4x + 16) - (2x^2 + 10) \right| dx$$

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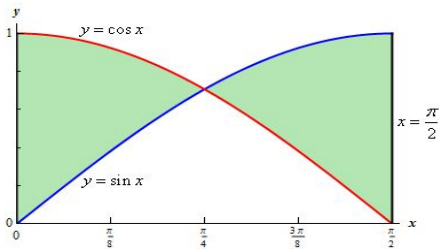
$$\int_{-2}^5 \left| (4x + 16) - (2x^2 + 10) \right| dx = \int_{-2}^{-1} (2x^2 + 10) - (4x + 16) dx +$$

$$\int_{-1}^3 ((4x + 16) - (2x^2 + 10)) dx + \int_3^5 (2x^2 + 10) - (4x + 16) dx = \dots$$

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Solution:



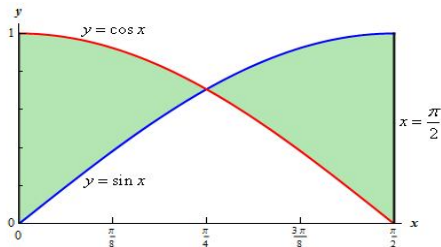
To find the intersection point of the curves $y = \cos x$ and $y = \sin x$ with $x \in [0, \frac{\pi}{2}]$, we solve the equation

$$\sin x = \cos x, \quad x \in [0, \frac{\pi}{2}].$$

Thus the intersection point is $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$.

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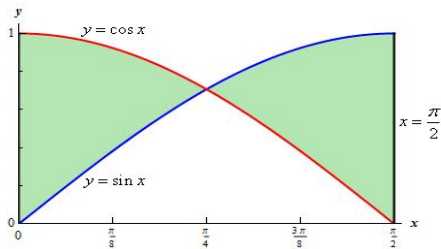
$$\sin x = \cos x, \quad x \in [0, \frac{\pi}{2}].$$

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$$\int_0^{\pi/2} |\cos x - \sin x| dx$$

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Thus the intersection point is $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$. Then the area of S is given by

$$\int_0^{\pi/2} |\cos x - \sin x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx = \dots$$

Remark:

If a region is bounded by curves with equations

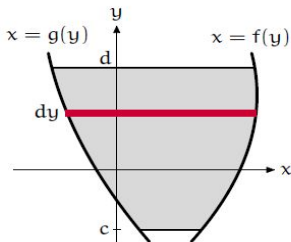
$$x = f(y), \quad x = g(y), \quad y = c, \quad \text{and} \quad y = d,$$

where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$. That is,

$$S = \{(x, y) : c \leq y \leq d; g(y) \leq x \leq f(y)\}.$$

Then the area of S is

$$A = \int_c^d (f(y) - g(y)) dy$$



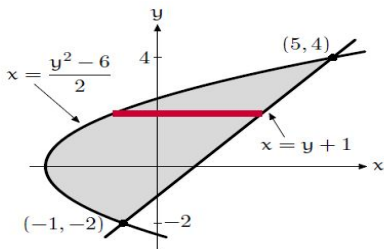
Example Find the area between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

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Solution: Considering x as a function of y , we have

$$x = y + 1; \quad x = \frac{y^2 - 6}{2}.$$

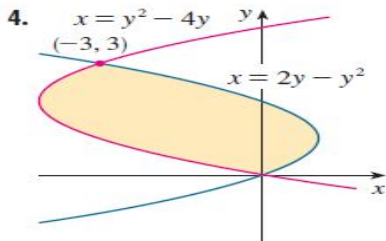
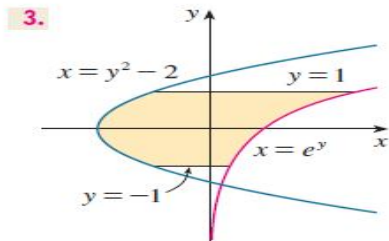
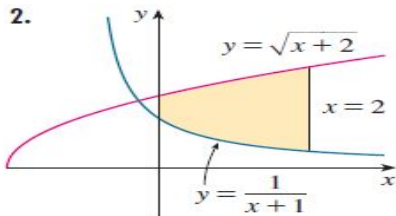
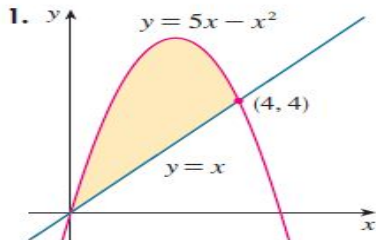
By solving $y + 1 = (y^2 - 6)/2$, we can deduce that the points of intersection are $(-1, -2)$ and $(5, 4)$.



$$S = \{(x, y) : -2 \leq y \leq 4; (y^2 - 6)/2 \leq x \leq y + 1\}.$$

The area is given by $\int_{-2}^4 (y + 1) - ((y^2 - 6)/2) dy = 18$.

1-4 ■ Find the area of the shaded region.



5.1 AREAS BETWEEN CURVES

5.1.3 AREA ENCLOSED BY PARAMETRIC CURVES

Suppose that f is continuous and $f(x) \geq 0$ on $[a, b]$. If the curve $y = f(x)$ is specified by parametric equations

$$x = \varphi(t), \quad y = \psi(t),$$

then the area of the region under the curve $y = f(x)$ from a to b is given by

$$S = \int_{\alpha}^{\beta} \psi(t) \varphi'(t) dt$$

where $\varphi(\alpha) = a$ and $\varphi(\beta) = b$.

5.1 AREAS BETWEEN CURVES

5.1.3 AREA ENCLOSED BY PARAMETRIC CURVES

Example: Compute the area of the region bounded by the x -axis and an arc of the cycloid

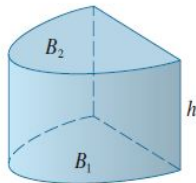
$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

Example: Compute the area of the region bounded by the ellipse
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

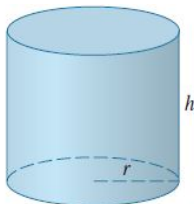
5.2 Volumes

5.2.1 Volumes by slicing

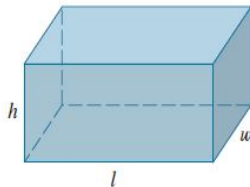
Recall that the volume of a right cylinder is Ah , where A is the area of the base and h is the height, measured perpendicular to the base.



(a) Cylinder
 $V = Ah$



(b) Circular cylinder
 $V = \pi r^2 h$

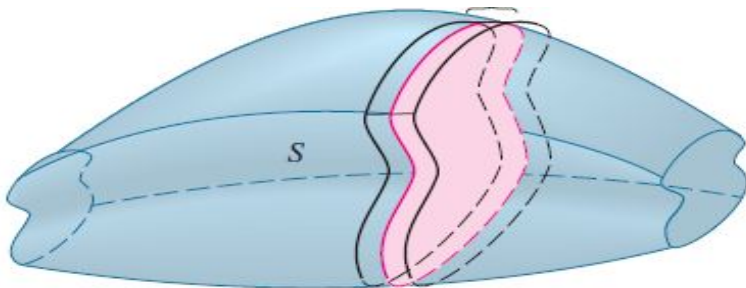


(c) Rectangular box
 $V = lwh$

5.2 Volumes

5.2.1 Volumes by slicing

Question: How to evaluate the volume of a solid S that isn't a cylinder?

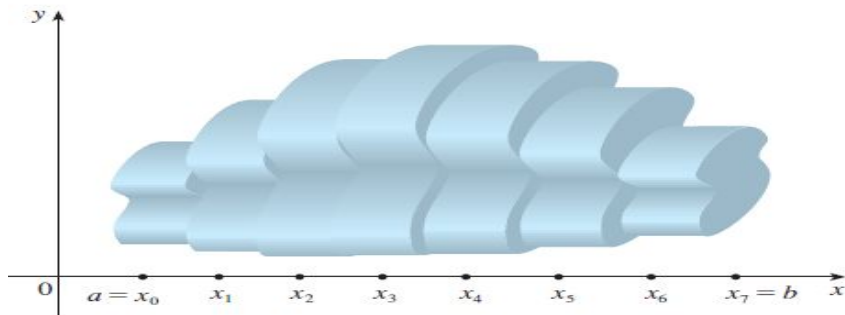


5.2 Volumes

5.2.1 Volumes by slicing

Answer:

We first cut S into pieces and approximate each piece by a cylinder. We estimate the volume of S by adding the volumes of the cylinders. We arrive at the exact volume of S through a limiting process in which the number of pieces becomes large.

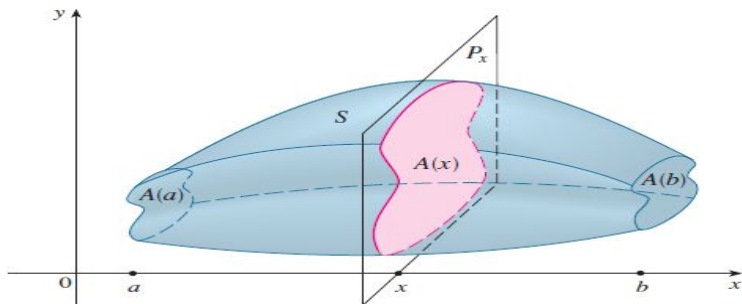


5.2 Volumes

5.2.1 Volumes by slicing

Evaluating the volume V of the solid S :

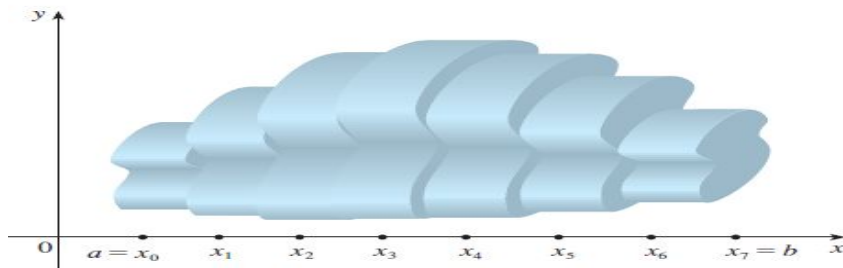
- We intersect S with a plane perpendicular to the x -axis and obtain a plane region that is called a **cross-section** of S . Let $A(x)$ be its area.



5.2 Volumes

5.2.1 Volumes by slicing

To compute the volume V of the solid S , we divide the solid into n vertical slices of equal width $\Delta x = (b - a)/n$.



If n is very large, then Δx is very small and the slices are very thin. In addition, if the function $A(x)$ is continuous, it doesn't change much in a short interval. So, the i^{th} slice is nearly a right cylinder of base area $A(x_{i-1})$ and height Δx . Therefore its volume V_i is approximated by $\Delta x \cdot A(x_{i-1})$.

5.2 VOLUMES

5.2.1 VOLUMES BY SLICING

Summing up, we obtain

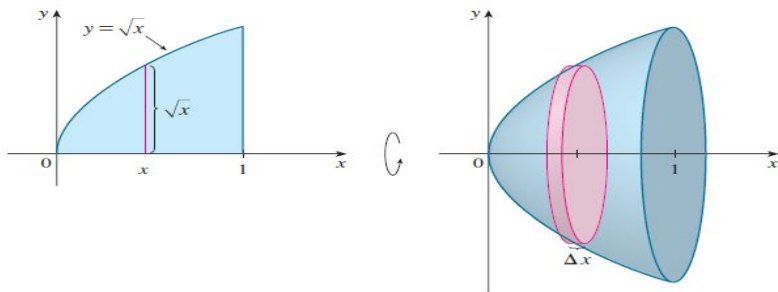
$$V = \sum_{i=1}^n V_i \approx \sum_{i=1}^n \Delta x A(x_{i-1}).$$

Definition

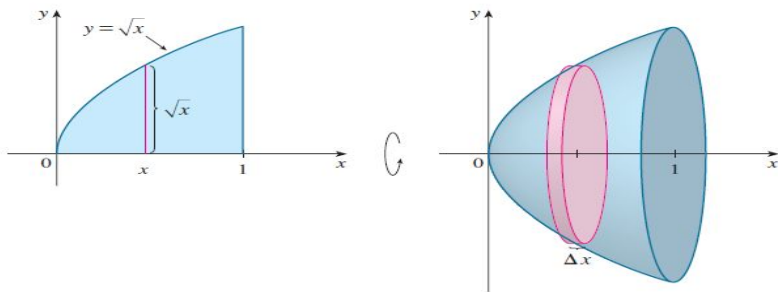
The volume V of a solid between $x = a$ and $x = b$ having cross-sectional area $A(x)$ at position x is

$$V = \int_a^b A(x) dx$$

Example: Find the volume of the solid obtained by rotating about the x -axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



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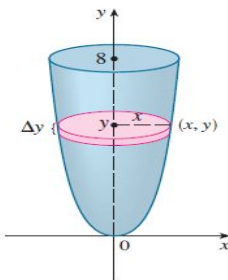
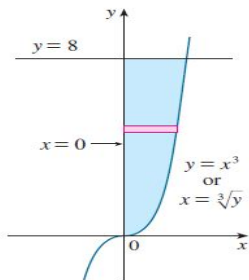
Solution: When we slice through the point x , we get a disk with radius $\sqrt{x}$. The area of this cross-section is

$$A(x) = \pi(\sqrt{x})^2 = \pi x.$$

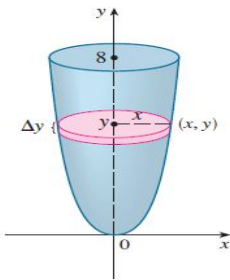
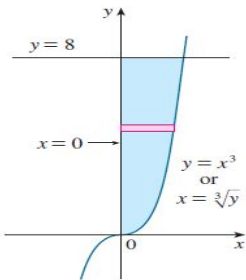
The solid lies between $x = 0$ and $x = 1$, so its volume is

$$V = \int_0^1 A(x) dx = \int_0^1 \pi x dx = \pi/2.$$

Example: Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y -axis.



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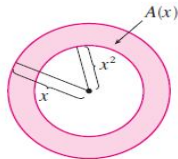
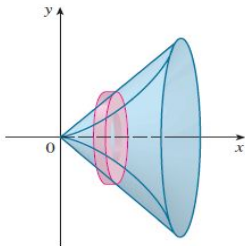
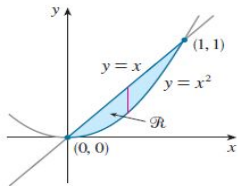
Solution: Because the region is rotated about the y -axis, it makes sense to slice the solid perpendicular to the y -axis and therefore to integrate with respect to y . If we slice at height y , we get a circular disk with radius x , where $x = \sqrt[3]{y}$. So the area of a cross-section through y is

$$A(y) = \pi(\sqrt[3]{y})^2.$$

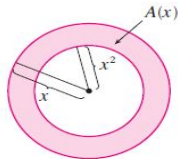
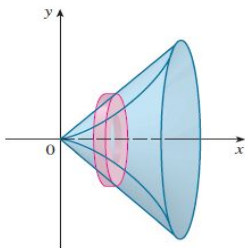
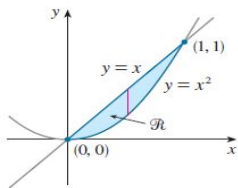
Because y varies from 0 to 8, the volume of the solid is given by

$$V = \int_0^8 A(y) dy = \int_0^8 \pi y^{2/3} dy = \frac{96\pi}{5}.$$

Example: The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x -axis. Find the volume of the resulting solid.



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Solution: A cross-section in the plane P_x has the shape of a washer (an annular ring) with inner radius x^2 and outer radius x . So the area of $A(x)$ is

$$A(x) = \pi(x^2 - x^4).$$

The volume of the solid is

$$\int_0^1 A(x) dx = \int_0^1 \pi(x^2 - x^4) dx = \frac{2\pi}{15}.$$

5.2 VOLUMES

5.2.2 SOLIDS OF REVOLUTION

More generally,

The volume of the solid generated by rotating the region
 $R = \{(x, y) : a \leq x \leq b, g(x) \leq y \leq f(x)\}$ *about the x-axis is*

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$$

Example Find the volume of the solid generated by rotating the region to the right of the y-axis and to the left of the curve $x = 2y - y^2$ about the y-axis.

5.2 VOLUMES

5.2.3 CYLINDRICAL SHELLS

Suppose that the region R bounded by

$$y = f(x) \geq 0, \quad y = 0, \quad x = a \geq 0 \quad \text{and} \quad x = b > a$$

is rotated about the y -axis to generate a solid of revolution.

The volume of the solid obtained by rotating the plane region $0 \leq y \leq f(x)$, $0 \leq a \leq x \leq b$ about the y -axis is

$$V = 2\pi \int_a^b xf(x)dx$$

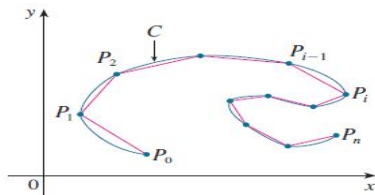
Example 2.5 A disk of radius a has center at the point $(b, 0)$, where $0 < a < b$. The disk is rotated about the y -axis to generate a *torus* (doughnut-shaped solid). Find its volume.

5.3 Arc length

Suppose that a curve C is described by the parametric equations

$$x = f(t), \quad y = g(t), \quad t \in [a, b].$$

We divide the parameter interval $[a, b]$ into n subintervals of equal width Δt . If $t_0, t_1, t_2, \dots, t_n$, are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \dots, P_n$ approximates C .



The length L of C is approximately the length of this polygon and the approximation gets better as we let n increase. Therefore, we define the length of C to be the limit of the lengths of these inscribed polygons

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1}P_i|.$$

5.3 Arc length

We have

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

where $\Delta x_i = x_i - x_{i-1}$; $\Delta y_i = y_i - y_{i-1}$. Thus

$$\Delta x_i = f(t_i) - f(t_{i-1}) \approx f'(t_i)(t_i - t_{i-1}) = f'(t_i)\Delta t.$$

and

$$\Delta y_i = g(t_i) - g(t_{i-1}) \approx g'(t_i)(t_i - t_{i-1}) = g'(t_i)\Delta t.$$

This gives,

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \approx (\sqrt{f'(t_i)^2 + (g'(t_i))^2})\Delta t.$$

Therefore,

$$L \approx \sum_{i=1}^n (\sqrt{f'(t_i)^2 + (g'(t_i))^2})\Delta t.$$

This is a Riemann sum for the function $\sqrt{f'(t)^2 + (g'(t))^2}$ and we thus get

$$L = \int_a^b (\sqrt{f'(t)^2 + (g'(t))^2}) dt.$$

5.3 Arc length

1 Arc Length Formula If a smooth curve with parametric equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b , then its length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

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Example: Find the length of the arc of the curve $x = t^2$, $y = t^3$, that lies between the points $(1, 1)$ and $(4, 8)$.

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Example: Find the length of the arc of the curve $x = t^2$, $y = t^3$, that lies between the points $(1, 1)$ and $(4, 8)$.

Solution: First we notice from the equations $x = t^2$ and $y = t^3$ that the portion of the curve between $(1, 1)$ and $(4, 8)$ corresponds to the parameter interval $1 \leq t \leq 2$. Thus the arc length formula gives

$$\begin{aligned} L &= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2} dt = \int_1^2 \sqrt{4 + 9t^2} t dt \\ &= \frac{1}{18} \int_1^2 \sqrt{4 + 9t^2} d(4 + 9t^2) = \frac{1}{27} (80\sqrt{10} - 13\sqrt{13}). \end{aligned}$$

5.3 ARC LENGTH

If we are given a curve with equation $y = f(x)$, $a \leq x \leq b$, then we can regard x as a parameter. Then parametric equations are $x = x$, $y = f(x)$, and Formula 1 becomes

2

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Similarly, if a curve has the equation $x = f(y)$, $a \leq y \leq b$, we regard y as the parameter and the length is

3

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$

5.3 ARC LENGTH

Example Find the length of the curve $y = x^4 + \frac{x^2}{32}$ from $x = 1$ to $x = 2$.

Example Find the arc length of $y = \cosh x$ over $[0, a]$. Then find the arc length over $[0, 2]$.

Example (a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, 1/2)$.

(b.) Use Simpsons Rule with $n = 10$ to estimate the arc length.

5.4 AVERAGE VALUE OF A FUNCTION

The average of the n numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ is given by

$$\frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = \frac{1}{n} \sum_{k=1}^n \alpha_k.$$



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Question: How do we compute the average value of a function $y = f(x)$, $a \leq x \leq b$?

5.4 AVERAGE VALUE OF A FUNCTION

Definition

If f is integrable on $[a, b]$, then the **average value** or **mean value** of f on $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

5.4 AVERAGE VALUE OF A FUNCTION

Example: Suppose the flow of blood at time t in a system is given by

$$F(t) = \frac{F_1}{(1 + \alpha t)^2}, \quad 0 \leq t \leq T,$$

where F_1 and α are constants. Find the average flow $\bar{F}$ on the interval $[0, T]$.

Example: Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

Example: A researcher estimates that t hours after midnight during a typical 24-hour period, the temperature in a certain northern city is $T(^{\circ}\text{C})$ where

$$T(t) = 3 - \frac{2}{3}(t - 13)^2, \quad 0 \leq t \leq 24.$$

What is the average temperature in the city between 6 A.M. and 4 P. M.?

Review

- 1) Related Rates
- 2) Maximum and Minimum Values
- 3) The Mean Value Theorem and its applications, Increasing and Decreasing Functions, The First Derivative Test, The Second Derivative Test, ...
- 4) Indeterminate Forms and l'Hospital's Rule
- 5) Optimization Problems
- 6) Indefinite integrals, Definite integrals: The Fundamental Theorem of Calculus, The Substitution Rule, Integration by Parts, Techniques of Integration...
- 7) Improper Integrals
- 8) Areas and Volumes